

SPACES OF REGULAR ABSTRACT MARTINGALES

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ABSTRACT. In [15, 10], the authors introduced and studied the space \mathcal{M}_r of regular martingales on a vector lattice and the space M_r of bounded regular martingales on a Banach lattice. In this note, we study these two spaces from the vector lattice point of view. We show, in particular, that these spaces need not be vector lattices. However, if the underlying space is order complete then \mathcal{M}_r is a vector lattice and M_r is a Banach lattice under the regular norm.

1. THE SPACE OF REGULAR MARTINGALES ON A VECTOR LATTICE

Let F be a vector lattice. A sequence (E_n) of positive projections on F such that $E_n E_m = E_{n \wedge m}$ is said to be a **filtration**. We will try to impose as few additional assumptions on the filtration as possible. A sequence $X = (x_n)$ in F is a **martingale** with respect to the filtration (E_n) if $E_n x_m = x_n$ whenever $m \geq n$. A sequence $X = (x_n)$ in F is a **supermartingale** if $E_n x_m \leq x_n$ whenever $m \geq n$ (note that in our definition we do not require that $E_n x_n = x_n$). We denote with $\mathcal{M} = \mathcal{M}(F, (E_n))$ the space of all martingales on F with respect to the filtration (E_n) . The space \mathcal{M} equipped with the coordinate-wise order is an ordered vector space and we denote with \mathcal{M}_+ the positive cone of \mathcal{M} . There is an extensive literature on abstract martingales on vector and Banach lattices, see, e.g., [5, 18, 11, 15, 10, 12, 8, 9, 6, 7]. For unexplained terminology on ordered vector and Banach spaces we refer the reader to [2, 3, 13].

The space of **regular martingales** is defined as follows:

$$\mathcal{M}_r = \mathcal{M}_r(F, (E_n)) = \{X_1 - X_2 : X_1, X_2 \in \mathcal{M}_+\}.$$

Equivalently, a martingale X is regular iff $\pm X \leq Y$ for some positive martingale Y . This definition is motivated by the definition of a regular

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operator. In this setting, it is a well known fact that the space of regular operators $\mathcal{L}_r(F)$ is itself a vector lattice when F is order complete. So it has been a natural conjecture that \mathcal{M}_r is a vector lattice whenever F is order complete. We will prove that this is indeed the case. This result improves [10, Theorem 2.3], which asserts that \mathcal{M}_r is a vector lattice (and is even order complete) provided that F is order complete and for every n the projection E_n is order continuous and $\text{Range } E_n$ is an order complete sublattice of F .

It has also been an open question whether \mathcal{M}_r is *always* a vector lattice. We will present an example to the contrary.

Theorem 1.1. *Let F be an order complete vector lattice and (E_n) a filtration on F . Then \mathcal{M}_r is an order complete vector lattice.*

Proof. Let \mathcal{A} be a subset of \mathcal{M}_r such that \mathcal{A} is bounded from above in \mathcal{M}_r . We will show that $\sup \mathcal{A}$ exists in \mathcal{M}_r . Let \mathcal{S} be the set of all supermartingales Y that dominate \mathcal{A} , that is, $X \leq Y$ for all $X \in \mathcal{A}$. By assumption, \mathcal{S} is non-empty. For every n , put

$$z_n = \inf \{y_n : Y = (y_k) \in \mathcal{S}\}.$$

We claim that $Z = (z_n)$ is a martingale and $Z = \sup \mathcal{A}$.

First, observe that $X \leq Z$ for each $X = (x_n) \in \mathcal{A}$. Indeed, for each n and each $Y = (y_n) \in \mathcal{S}$ we have $x_n \leq y_n$, so that $x_n \leq z_n$.

Next, observe that Z is a supermartingale. Let $m \geq n$, then for every $Y \in \mathcal{S}$ we have $z_m \leq y_m$, so that $E_n z_m \leq E_n y_m \leq y_n$. It follows that $E_n z_m \leq z_n$.

Next, we will show that Z is, in fact, a martingale. Fix $k \in \mathbb{N}$ and define $Y = (y_n)$ as follows:

$$Y = (E_1 z_k, E_2 z_k, \dots, E_{k-1} z_k, z_k, z_{k+1}, z_{k+2}, \dots).$$

We claim that Y is a supermartingale. Indeed, let $n \leq m$.

$$\begin{aligned} \text{If } k \leq n \quad & \text{then} \quad E_n y_m = E_n z_m \leq z_n = y_n; \\ \text{if } m < k \quad & \text{then} \quad E_n y_m = E_n E_m z_k = E_n z_k = y_n; \\ \text{if } n < k \leq m \quad & \text{then} \quad E_n y_m = E_n z_m = E_n E_k z_m \leq E_n z_k = y_n. \end{aligned}$$

Next, note that $X \leq Y$ for each $X = (x_n) \in \mathcal{A}$. Indeed, if $n \geq k$ then $y_n = z_n \geq x_n$. If $n < k$ then $x_k \leq z_k$ implies $E_n x_k \leq E_n z_k$, so that $x_n \leq y_n$.

This yields that $Y \in \mathcal{S}$, so that $Z \leq Y$. It follows that for every $n < k$ we have $z_n \leq y_n = E_n z_k$, so that $z_n \leq E_n z_k$. Therefore, $z_n = E_n z_k$ for all n and k with $n < k$. Also note that $E_n z_n = E_n E_n z_{n+1} = E_n z_{n+1} = z_n$ for every n . Thus, Z is a martingale and $X \leq Z$ for

each $X \in \mathcal{A}$. Clearly, every martingale Y dominating \mathcal{A} is in \mathcal{S} and, therefore, $Z \leq Y$. Hence, $Z = \sup \mathcal{A}$.

Let $X \in \mathcal{M}_r$. Applying the previous argument with $\mathcal{A} = \{\pm X\}$, we conclude that $|X|$ exists. It follows that \mathcal{M}_r is a vector lattice. By the preceding computation, it is order complete. \square

Example 1.2. \mathcal{M}_r need not be a vector lattice. Let $F = c$. For each n , define $E_n: F \rightarrow F$ as follows: if $x = (\alpha_i)$ we put

$$E_n x = \left(\alpha_1, \dots, \alpha_{3n}, \frac{\alpha_{3n+1} + \alpha_{3n+2}}{2}, \frac{\alpha_{3n+1} + \alpha_{3n+2}}{2}, \alpha_{3n+3}, \right. \\ \left. \frac{\alpha_{3n+4} + \alpha_{3n+5}}{2}, \frac{\alpha_{3n+4} + \alpha_{3n+5}}{2}, \alpha_{3n+6}, \dots \right).$$

It is easy to see that (E_n) is a filtration on c and each E_n is order continuous. It is a **dense** filtration in the sense that $\bigcup_{n=1}^{\infty} \text{Range } E_n$ is dense in F . Define (x_n) as follows:

$$x_n = \underbrace{(1, -1, 0, 1, -1, 0, \dots, 1, -1, 0, 0, 0, \dots)}_{3n}.$$

Note that $X = (x_n)$ is a martingale with respect to (E_n) ; it is regular because $\pm x_n \leq 1$ for every n , where 1 is the constant one sequence. We will write $x_{n,i}$ for the i -th coordinate of x_n . We claim that X has no modulus in \mathcal{M}_r . Indeed, suppose that $\pm X \leq Y$ for some martingale Y , $Y = (y_n)$. For each n we have $y_n \geq \pm x_n$, so that

$$y_n \geq \underbrace{(1, 1, 0, 1, 1, 0, \dots, 1, 1, 0, 0, 0, \dots)}_{3n} = u_n.$$

It follows that

$$y_1 = E_1 y_n \geq E_1 u_n = u_n.$$

Since n is arbitrary, it follows that $y_{1,3k+1} \geq 1$ and $y_{1,3k+2} \geq 1$ for every k . Since y_1 is an element of c , there is k_0 such that $y_{1,3k_0} > 0$. Define a martingale $Z = (z_n)$ as follows: for every n and i , put $z_{n,i} = y_{n,i}$ except when $i = 3k_0$, in this case put $z_{n,3k_0} = 0$ (for every n). It is easy to see that Z is a martingale and $\pm X \leq Z < Y$. It follows that X has no modulus.

2. KRICKEBERG'S FORMULA

Once again using the analogy with regular operators, we recall that if F is an order complete vector lattice then $\mathcal{L}_r(F)$ is a vector lattice and the lattice operations on $\mathcal{L}_r(F)$ are given by the Riesz-Kantorovich formula. There is a similar formula for lattice operations on \mathcal{M}_r . Let $X = (x_n)$ be a martingale. It has been observed in the literature that

that the modulus $|X|$ of a regular martingale $X = (x_n)$ often satisfies the following identity:

$$|X|_n = \sup_{m \geq n} E_n |x_m|.$$

For classical martingales, this identity goes back to Krickeberg's decomposition (see i.e., [14, p. 32]); in the following we will refer to it as **Krickeberg's formula**. If the Krickeberg's formula is valid for F , that is, if the modulus of every martingale in $\mathcal{M}_r(F, (E_n))$ is given by the Krickeberg's formula, then, clearly, \mathcal{M}_r is a vector lattice and the other lattice operations are given by similar formulae; see, e.g., [15, Theorem 7].

In the following proposition, we summarize several cases where \mathcal{M}_r is a vector lattice and the lattice operations are given by Krickeberg's formula; it extends [10, Theorem 2.3], [15, Proposition 11], and [8, Proposition 4].

Proposition 2.1. *Let F be a vector lattice and (E_n) a filtration on F . Suppose that any of the following hold.*

- (1) *F is order complete and each E_n is order continuous;*
- (2) *F is Archimedean and E_n is of finite rank for each n ;*
- (3) *E_n is a lattice homomorphism for each n .*

Then \mathcal{M}_r is a vector lattice and the lattice operations are given by the Krickeberg's formula.

Proof. Let $X = (x_n)$ be a martingale such that $\pm X \leq Y$ for some positive martingale $Y = (y_n)$. For a fixed n , the sequence $(E_n |x_m|)_{m=n}^{\infty}$ is increasing in m , bounded below by $|x_n|$ and above by y_n . Indeed, if $n \leq m$ then

$$E_n |x_m| = E_n |E_m x_{m+1}| \leq E_n E_m |x_{m+1}| = E_n |x_{m+1}|,$$

$$|x_n| = |E_n x_n| \leq E_n |x_n|, \quad \text{and} \quad E_n |x_m| \leq E_n y_m = y_n.$$

(1) Since F is order complete, $\sup_{m \geq n} E_n |x_m|$ exists for every n . Denote it z_n and put $Z = (z_n)$. Clearly, $\pm X \leq Z \leq Y$. Note that Z is a martingale: for every $k \leq n$, since E_k is order continuous, we have

$$E_k z_n = E_k \left(\sup_{m \geq n} E_n |x_m| \right) = \sup_{m \geq n} E_k E_n |x_m| = \sup_{m \geq n} E_k |x_m| = z_k.$$

It follows that $Z = |X|$. Note that Z is given by the Krickeberg's formula.

(2) Let $H_n = \text{Range } E_n$ and $H_n^+ = F^+ \cap H_n$. Since H_n is finite-dimensional, we may view it as an ordered Banach space. Note that $(E_n |x_m|)_{m=n}^{\infty}$ and y_n are in H_n^+ . Since F is Archimedean and H_n

is finite-dimensional, H_n^+ is closed (in H_n) by [3, Corollary 3.4] and normal by [3, Lemma 3.1]. It follows from [3, Theorem 2.45] that $\lim_m E_n|x_m| = z_n$ in H_n . Since H_n^+ is closed, it follows from $|x_n| \leq E_n|x_m| \leq y_n$ that $|x_n| \leq z_n \leq y_n$.

Repeating this process for every n , we produce a sequence $Z = (z_n)$ in F_+ . To show that Z is a martingale, let $k \leq n$. Since E_k is a continuous operator on H_n , we have

$$E_k z_n = E_k \left(\lim_{m \rightarrow \infty} E_n|x_m| \right) = \lim_{m \rightarrow \infty} E_k E_n|x_m| = \lim_{m \rightarrow \infty} E_k|x_m| = z_k,$$

where the limit is taken in H_n . It follows from $\pm X \leq Z \leq Y$ that $Z = |X|$.

To verify Krickeberg's formula, it suffices to show that $z_n = \sup_m E_n|x_m|$. It follows from [3, Lemma 2.3(4)] that $z_n = \sup_m E_n|x_m|$ in H_n . Let $a \in F$ such that $E_n|x_m| \leq a$ for all $m \geq n$. Let G be the subspace of F spanned by H_n and a . Again, we may view it as an ordered Banach space with closed positive cone $G_+ = F_+ \cap G$; H_n is a closed subspace of G . Hence, we still have $\lim_m E_n|x_m| = z_n$ in G . Applying [3, Lemma 2.3(4)] to G , we conclude that $z_n = \sup_m E_n|x_m|$ in G , and, therefore, $z_n \leq a$.

(3) It is easy to see that the sequence $(|x_n|)$ is a martingale and is the modulus of X . Also, for every fixed n and every $m \geq n$ we have $E_n|x_m| = |E_n x_m| = |x_n|$, so that Krickeberg's formula is valid. \square

It is an open problem whether the modulus of an operator is always given by the Riesz-Kantorovich formula, see [3, p. 59] for details. Similarly, it has been a natural conjecture that Krickeberg's formula is always valid whenever \mathcal{M}_r is a vector lattice. However, we will present a counterexample to the contrary. Our example will be based on [8, Example 6], which we outline here for convenience of the reader.

Example 2.2. ([8]) Let $F = \mathbb{R}^{\mathbb{N}}$. For $n = 0, 1, 2, \dots$, define E_n via

$$E_n((a_i)) = \left(a_1, a_2, \dots, a_{2n}, \frac{a_{2n+1} + a_{2n+2}}{2}, \frac{a_{2n+1} + a_{2n+2}}{2}, \right. \\ \left. \frac{a_{2n+3} + a_{2n+4}}{2}, \frac{a_{2n+3} + a_{2n+4}}{2}, \dots \right)$$

Let $X = (x_n)_{n=0}^{\infty}$ where

$$x_n = \left(\underbrace{-1, 1, \dots, -1, 1}_{2n}, 0, 0, \dots \right)$$

(we take $x_0 = 0$). It is easy to see that (E_n) is a filtration on F and X is a martingale with respect to (E_n) .

Example 2.3. \mathcal{M}_r is a vector lattice, yet Krickeberg's formula fails. Let $F = \ell_\infty$; let (E_n) and $X = (x_n)$ be as in Example 2.2, $n = 0, 1, \dots$. Let $\varphi: F \rightarrow \mathbb{R}$ be a Banach limit. Put $P = \varphi \otimes 1$. It is easy to see that P is a rank-one projection and that $E_n P = P$ for every n . It follows that the sequence $(PE_0, E_0, E_1, E_2, \dots)$ is a filtration on F and the sequence $X = (x_0, x_0, x_1, x_2, \dots)$ is a martingale with respect to this filtration. Note that \mathcal{M}_r is a vector lattice by Theorem 1.1.

We claim that $|X| = (1, 1, 1, 1, \dots)$. Indeed, it is easy to see that the sequence $(1, 1, 1, 1, \dots)$ is a martingale which dominates $\pm X$. Now suppose $\pm X \leq Y$ for some martingale Y . Put $Y = (z, y_0, y_1, \dots)$. For every $n \geq 0$ and $m \geq n$ we have $y_m \geq |x_m|$, so that $y_n = E_n y_m \geq E_n |x_m|$. Note that

$$E_n |x_m| = |x_m| = (\underbrace{1, 1, \dots, 1}_{2m}, 0, 0, \dots).$$

This yields $y_n \geq 1$ for every $n \geq 0$. It follows from $y_0 \geq 1$ that $z = PE_0 y_0 \geq 1$. Hence, $Y \geq (1, 1, 1, 1, \dots)$. Therefore, $|X| = (1, 1, 1, 1, \dots)$.

However, Krickeberg's formula for the initial term gives $\sup_m PE_0 |x_m| = 0$ instead of 1.

3. THE SPACE OF REGULAR BOUNDED MARTINGALES ON A BANACH LATTICE

We say that (E_n) is **uniformly bounded** if $\sup_n \|E_n\| < +\infty$; we say that (E_n) is **contractive** if $\|E_n\| \leq 1$ for every n . A martingale $X = (x_n)$ in $\mathcal{M}(F, (E_n))$ is said to be **bounded** if its **martingale norm** defined by $\|X\| = \sup_n \|x_n\|$ is finite. We denote by $M = M(F, (E_n))$ the space of all bounded martingales on F with respect to the filtration (E_n) . It is easy to see that M is a closed subspace of $\ell_\infty(F)$; hence M is a Banach space. It can be easily verified that the martingale norm is **monotone**, i.e., $0 \leq X \leq Y$ implies $\|X\| \leq \|Y\|$. The space of regular bounded martingales is the following subspace of M :

$$M_r = M_r(F, (E_n)) = \{X_1 - X_2 : X_1, X_2 \in M_+\}.$$

Again, one may expect similarities with the well-known theory of regular operators; see, e.g., [16, 17]. It is well known that every regular operator on a Banach lattice is bounded and the space of regular operators on an order complete Banach lattice is a Banach lattice under the *regular norm*. We will prove that if F is order complete then M_r is a Banach lattice under the regular norm. We will show that, in contrast to the setting of regular operators, in general $M_r \neq \mathcal{M}_r$. Furthermore,

F is a KB-space iff F is order continuous and every bounded martingale with respect to every uniformly bounded filtration is regular.

Example 3.1. *Positive unbounded martingale on ℓ_1 .* For any $0 \leq \alpha \leq 1$, define $P_\alpha = \begin{bmatrix} 0 & 0 \\ \alpha & 1 \end{bmatrix}$. It is easy to see that P_α is a positive projection onto e_2 , and P_α is a contraction when viewed as an operator on ℓ_1^2 . Define a filtration on ℓ_1 as follows.

$$E_1 = \begin{bmatrix} 0 & 0 & & & \\ 1 & 1 & & & \\ & & 0 & 0 & \\ & & \frac{1}{2} & 1 & \\ & & & & 0 & 0 \\ & & & & \frac{1}{4} & 1 \\ & & & & & \ddots \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & 0 & 0 & \\ & & \frac{1}{2} & 1 & \\ & & & & 0 & 0 \\ & & & & \frac{1}{4} & 1 \\ & & & & & \ddots \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & 1 & 0 & \\ & & 0 & 1 & \\ & & & & 0 & 0 \\ & & & & \frac{1}{4} & 1 \\ & & & & & \ddots \end{bmatrix}, \text{ etc.}$$

It is easy to see that this is a filtration $\|E_n\| = 1$. Further, define

$$\begin{aligned} x_1 &= (0, 1, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, \dots), \\ x_2 &= (1, 0, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, \dots), \\ x_3 &= (1, 0, 1, 0, 0, \frac{1}{4}, 0, \frac{1}{8}, \dots), \\ x_4 &= (1, 0, 1, 0, 1, 0, 0, \frac{1}{8}, \dots), \\ &\text{etc.} \end{aligned}$$

It can be easily verified that (x_n) is a positive martingale with respect to the filtration (E_n) , but $\|x_n\| > n - 1$, so this martingale is unbounded.

On M_r , we define the following so called **regular norm**:

$$\|X\|_r = \inf \{ \|Y\| : Y \in M_+, Y \geq \pm X \}.$$

We claim that the space $(M_r, \|\cdot\|_r)$ is a Banach space. We will prove a more general result for ordered Banach spaces. In particular this result is known to be true for the space of regular operators and the space generated by positive compact operators ([4], Proposition 2.2).

Theorem 3.2. *Suppose that $(X, \|\cdot\|)$ is an ordered normed space with a closed cone X_+ and a monotone norm. Then the following formula defines a norm on $X_r = X_+ - X_+$:*

$$\|x\|_r = \inf \{ \|y\| : y \in X_+, y \geq \pm x \}.$$

For every $z \in X_r$, we have $\|z\| \leq 2\|z\|_r$. Moreover, if X is complete then $(X, \|\cdot\|_r)$ is complete and if, in addition, $X = X_r$ then $\|\cdot\|$ and $\|\cdot\|_r$ are equivalent. If X_r is a vector lattice then $\|x\|_r = \||x|\|$ for all $x \in X_r$.

Proof. It is easy to see that $\|\cdot\|_r$ is positively homogeneous. To verify the triangle inequality, let $u, v \in X_r$, take any $\varepsilon > 0$, and find $x, y \in X_+$ such that $-x \leq u \leq x$ and $-y \leq v \leq y$, $\|x\| \leq \|u\|_r + \varepsilon$, and $\|y\| \leq \|v\|_r + \varepsilon$. It follows that $-(x+y) \leq u+v \leq x+y$, so that

$$\|u+v\|_r \leq \|x+y\| \leq \|x\| + \|y\| \leq \|u\|_r + \|v\|_r + 2\varepsilon.$$

This yields $\|u+v\|_r \leq \|u\|_r + \|v\|_r$.

Fix $z \in X_r$. Let $x \in X_+$ be such that $-x \leq z \leq x$. Then $0 \leq x \pm z \leq 2x$. It follows that

$$\|z\| = \left\| \frac{1}{2}(x+z) - \frac{1}{2}(x-z) \right\| \leq \frac{1}{2}\|x+z\| + \frac{1}{2}\|x-z\| \leq \frac{1}{2}\|2x\| + \frac{1}{2}\|2x\| = \|x\|.$$

Taking the infimum over all such x , we get $\|z\| \leq 2\|z\|_r$. In particular, if $\|z\|_r = 0$ then $\|z\| = 0$ and, therefore, $z = 0$.

Now suppose that $(X, \|\cdot\|)$ is complete and show that $(X_r, \|\cdot\|_r)$ is complete. Let (z_n) be a $\|\cdot\|_r$ -Cauchy sequence in X_r . Note that since $\|\cdot\| \leq 2\|\cdot\|_r$, the sequence is also Cauchy in the original norm, hence $z_n \xrightarrow{\|\cdot\|} z$ for some $z \in X$. It suffices to show that $z \in X_r$ and some subsequence of (z_n) converges to z in $\|\cdot\|_r$ because, in this case, the entire sequence would still converge to z in $\|\cdot\|_r$.

Without loss of generality, passing to a subsequence, we may assume that for every n and every $k \geq n$ we have $\|z_n - z_k\|_r \leq \frac{1}{3^n}$. For each n , $z_{n+1} - z_n \in X_r$, so we can find $x_n \in X_+$ such that $-x_n \leq z_{n+1} - z_n \leq x_n$ and $\|x_n\| \leq \frac{1}{2^n}$. Fix m . It follows from $z_n \xrightarrow{\|\cdot\|} z$ that $z - z_m = \sum_{n=m}^{\infty} (z_{n+1} - z_n)$, where the series converges in $\|\cdot\|$. Note that

$$\sum_{n=m}^k (z_{n+1} - z_n) \leq \sum_{n=m}^k x_n$$

for every $k > m$ and $\sum_{n=m}^{\infty} x_n$ converges in $\|\cdot\|$. Since X_+ is closed, $z - z_m \leq \sum_{n=m}^{\infty} x_n$. Similarly, $-(z - z_m) \leq \sum_{n=m}^{\infty} x_n$. It follows that

$$\|z - z_m\|_r \leq \left\| \sum_{n=m}^{\infty} x_n \right\| \leq \sum_{n=m}^{\infty} \|x_n\| \leq \frac{1}{2^{m-1}} \rightarrow 0.$$

If, moreover, $X = X_r$, we have that $\|\cdot\|$ and $\|\cdot\|_r$ are two complete norms on the same space with one of them dominating the other; it follows that the norms are equivalent.

Suppose that X_r is a vector lattice and $x \in X_r$. It follows from $|x| \geq \pm x$ that $\||x|\| \geq \|x\|_r$. On the other hand, if $\pm x \leq y$ for some $y \in X_+$ then $|x| \leq y$ and the monotonicity of norm yields $\||x|\| \leq \|y\|$, hence $\||x|\| \leq \|x\|_r$. \square

Corollary 3.3. *Let F be a Banach lattice and (E_n) a filtration on F . The space $(M_r, \|\cdot\|_r)$ is a Banach space and $\|X\| \leq \|X\|_r$ for every $X \in M_r$. If M_r is a vector lattice then $\|X\|_r = \||X|\|$ for all $X \in M_r$.*

Proof. Applying Theorem 3.2 to M , we conclude that $(M_r, \|\cdot\|_r)$ is a Banach space and that if M_r is a vector lattice then $\|X\|_r = \||X|\|$ for all $X \in M_r$. For $Y \in M_+$ such that $\pm X \leq Y$, by the definition of the martingale norm we have $\|X\| \leq \|Y\|$. It follows that $\|X\| \leq \|X\|_r$. \square

The regular norm may coincide with the martingale norm on M_r . We recall here that a Banach lattice F is said to have the **Fatou property** if $0 \leq x_\alpha \uparrow x$ implies $\|x_\alpha\| \rightarrow \|x\|$. Dual and order continuous Banach lattices enjoy the Fatou property; see, e.g., [13, p. 96] or [1, p. 65].

Proposition 3.4. *Let F be a Banach lattice with the Fatou property and (E_n) a contractive filtration on F . If M_r is a vector lattice and Krickeberg's formula is valid on M_r then $\|X\| = \|X\|_r$ for every $X \in M_r$.*

Proof. Let $X = (x_n) \in M_r$ and $Z = (z_n) = |X|$. By Corollary 3.3, $\|X\|_r = \|Z\| \geq \|X\|$. We need to show that $\|X\| \geq \|Z\|$. By Krickeberg's formula, for each $n \in \mathbb{N}$ we have $E_n|x_m| \uparrow z_n$ (in m). The Fatou property yields $\|z_n\| = \sup_m \|E_n|x_m|\| \leq \sup_m \|x_m\| = \|X\|$, hence $\|Z\| \leq \|X\|$. \square

The following is the main result of our paper. Note that in view of Proposition 3.4 this result extends Theorem 13 in [15]. Recall that if F is order complete then \mathcal{M}_r is a vector lattice by Theorem 1.1.

Theorem 3.5. *Let F be an order complete Banach lattice and (E_n) a filtration on F . Then the space M_r is an ideal of \mathcal{M}_r and a Banach lattice under the regular norm.*

Proof. Note first that if $X \in M_r$ and $Y \in \mathcal{M}_r$ such that $0 \leq Y \leq X$ then $Y \in M_r$. Let $X \in M_r$. Then there exists $Y \in M_r$ such that $Y \geq \pm X$. By Theorem 1.1, $|X|$ exists in \mathcal{M}_r . Clearly, $|X| \leq Y$ and, therefore, $|X| \in M_r$. Hence, M_r is a vector lattice and an ideal of \mathcal{M}_r .

By Corollary 3.3, $(M_r, \|\cdot\|_r)$ is a Banach lattice. \square

Example 3.6. M_r need not be a Banach lattice under the martingale norm. Let $F = \ell_\infty$, equipped with following equivalent norm:

$$\|(a_i)\| = \|(a_i)\|_\infty + \limsup |a_i|$$

It can be easily verified that $(F, \|\cdot\|)$ is a Banach lattice. Let (E_n) and $X = (x_n)$ be as in Example 2.2. Note that each E_n is order continuous. Clearly, $M = M_r$. By Theorem 3.5, M_r is a Banach lattice under $\|\cdot\|_r$ and an ideal in \mathcal{M}_r . For each n , we have

$$E_n |x_m| = |x_m| = \underbrace{(1, 1, \dots, 1, 1, 0, \dots)}_{2m} \uparrow 1,$$

By Proposition 2.1(1), lattice operations are given by Krickeberg's formula. It follows that the modulus of X is the constant martingale $|X|_n = 1$. We have $\||X|\| = 2$ and $\|x_n\| = 1$ for each n , thus $1 = \|X\| < \||X|\|$, so that M_r fails to be a Banach lattice under the martingale norm.

Finally, we study under which conditions we have $M = M_r$. Recall that a vector lattice F has a **strong unit** e whenever $F = \bigcup_{n=1}^\infty [-ne, ne]$. A Banach lattice F has an **order continuous norm** whenever $x_\alpha \downarrow 0$ implies $x_\alpha \rightarrow 0$; F is a **KB-space** if it does not contain a sublattice isomorphic to c_0 .

Proposition 3.7. *Let F be a Banach lattice with a strong unit e and (E_n) a uniformly bounded filtration on F . Then $M = M_r$. If, in addition, F is order complete then M is a vector lattice with a strong unit.*

Proof. It is known that the original norm of F is equivalent to the norm $\|\cdot\|_\infty$ generated by e ; see, e.g., [2, p. 194]. In particular, there exists $C > 0$ such that $|x| \leq C\|x\|e$ for every $x \in F$.

For each n , put $y_n = E_n e$. Clearly, $Y = (y_n)$ is a bounded positive martingale. Let $X \in M$. Then for every n we have $\pm x_n \leq C\|x_n\|e \leq C\|X\|e$. Applying E_n , we get $\pm x_n \leq C\|X\|y_n$. It follows that $\pm X \leq C\|X\|Y$, so that X is regular. Hence, $M = M_r$. If, in addition, F is order complete then M is a vector lattice by Theorem 3.5. It follows from $\pm X \leq C\|X\|Y$ that Y is a strong unit in M . \square

Theorem 3.8. *Let F be a Banach lattice with an order continuous norm. Then the following are equivalent.*

- (1) F is a KB-space;
- (2) $M = M_r$ for every uniformly bounded filtration (E_n) on F ;
- (3) M is a vector lattice for any uniformly bounded filtration (E_n) on F .

Proof. (3) \Rightarrow (2) trivially. (2) \Rightarrow (3) by Theorem 3.5

(1) \Rightarrow (3) The proof is similar to that of [15, Theorem 7]. Let $X = (x_n) \in M$. Fix some $n \in \mathbb{N}$. The sequence $(E_n|x_m|)_{m=n}^\infty$ is increasing and norm bounded. Since F is a KB-space, it follows that $z_n = \lim_m E_n|x_m|$ exists; then Z is a martingale and $Z = |X|$.

(2) \Rightarrow (1) Suppose that F is not a KB-space. Then c_0 is lattice embeddable in F . Without loss of generality, we can assume that c_0 is a closed sublattice of F . Let (E_n) and $X = (x_n)$ be as in Example 2.2; view (E_n) as a (uniformly bounded) filtration on c_0 and X as a martingale in c_0 .

By [13, Corollary 2.4.3], there exist a positive projection $P : F \rightarrow c_0$. It is easy to see that (PE_n) is again a uniformly bounded filtration on F and X is a bounded martingale with respect to it. We claim that X is not regular. Indeed, suppose that $\pm X \leq Z$ for some positive martingale $Z = (z_n)$ in F . Then $\pm x_n \leq z_n$ for every n yields $|x_n| \leq z_n$, so that $E_0|x_n| = PE_0|x_n| \leq PE_0z_n = z_0$. It follows that the increasing sequence $(E_0|x_n|)$ is bounded above in F , hence it converges in F because F is order continuous. Therefore, $(E_0|x_n|)$ converges in c_0 , which is clearly false. \square

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